There are alternate definitions for gradient, divergence and curl that involve a limiting process of the ratio of integral over enclosing surface to integral over enclosed volume as the volume approaches zero. The differential surface normal vector multiplies the scalar function for gradient. For divergence, the inner product of differential surface normal and function vector is used. For curl, the cross product of differential surface normal and function vector is used.

If we back down division algebras to the Quaternions, the forms for gradient, divergence and curl are all present in the singular Quaternion product of differential operator and function Quaternion. A natural extension of the integral definitions for gradient, divergence and curl mentioned above would be to first replace the 3D differential surface normal vector with the 4D Quaternion differential surface normal. Next form a singular definition for differentiation that is the limit as the enclosed 4D volume approaches zero for the ratio of integral of the product of Quaternion differential surface normal and Quaternion function divided by the integral of the enclosed 4d differential volume. This form covers the traditional forms for gradient, divergence and curl as an ensemble, which is why I call it the Ensemble Derivative.

It is a straight shot from this Quaternion definition to one for the Octonions. We just need to substitute the 8D definition for differential surface normal, Octonion function, Octonion product and 8D differential volume.

The beauty of the integral definition for differentiation is the ability to easily move to a description using an alternate variable set. If a functional relationship exists between the variable sets, Jacobian formalism can be used to cast the differential surface normal and differential volume in terms of the new set of variables, Jacobians, and cofactors of the Jacobian matrix.

I take the integral definition for differentiation as its fundamental definition, not just an alternate form. By defining differentiation fundamentally using a diffeomorphism between the intrinsic system attached to the algebra directly and an alternate, the transformation properties for differential equations are intrinsic to the definition itself, not an afterthought or something tacked on. There is more here than say, a conversion from rectilinear coordinates to spherical-polar, although this is certainly covered. When the system is a hypercomplex one, as with the Quaternions and Octonions, the door is open to allow one of the variables to represent time, and the diffeomorphism to represent a velocity transformation. As with any curvilinear system, the velocity components present in the Jacobian matrix may freely be functions of time, in other words accelerated frames of reference will be covered as simply curvilinear in time.

The task is to define the Ensemble Derivative form at a single point in the coordinate space. The volume in question always includes this single point as an interior point. The surface in question always surrounds the single point without ever contacting it. The limit process allows the surface to come arbitrarily close to the single point.

Since we have an ASCII character format here, please take d/dri as the i'th partial derivative with respect to the fundamental basis coordinate system directly attached to the Octonion basis units. In other words, ri ui sum i from 0 to 7 is the fundamental Octonion position vector. Take d/dvi as the i'th partial derivative with respect to the diff-morphed system of coordinates. Then dri and dvi are the i'th differentials of their respective systems.

The volume always includes the point in question as an interior point. The differential volume can therefore be expressed in the v system as

J dv0 dv1 dv2 dv3 dv4 dv5 dv6 dv7

Here J is the Jacobian dr/dv of the diffeomorphism from r to v, evaluated at the point where we wish to define the derivative at by mean value arguments. Since it is evaluated at a single point, its variation about the coordinate neighborhood of this point is not in issue. This Jacobian thus can thus be brought outside the integral as a constant scaling factor 1/J on the eventual form for the Ensemble Differentiation.

The simplest form for the differential surface normal is the Octonion form (summation over all i)

dNi = J dvi/drj uj dv0 dv1 dv2 dv3 dv4 dv5 dv6 dv7 / dvi

Unlike the Jacobian in the differential volume element, both J and dvi/drj here are evaluated off the single point at which we wish to define the Ensemble form. Their variation in the coordinate neighborhood of the point in question is very much an issue in the definition of the Ensemble Derivative form.

If we take F(v) as the Octonion function to differentiate, the limit process will yield the following for the Ensemble derivative E of F(v)

E(F) = 1/J d/dvi [J dvi/drj uj*F] sum ij

Here "*" is Octonion multiplication of F by basis unit uj as defined by the algebra representation of choice. We may write F in terms of its connection to the fundamental Octonion basis units as

F(v) = Fk drl/dvk ul

Then the Ensemble form may be written as

E(F) = 1/J d/dvi [J dvi/drj drl/dvk Fk] uj ul   sum ijkl

The fundamental basis units are constants, so may be brought outside the differentiation.

The Ensemble Derivative with respect to v on F(v) can be associated with its equivalent function G(r) simply by equating r to v. Then the Jacobian is real unity, and dvi/drj is non zero unity only for i=j, and drl/dvk is non zero unity only for k=l. The Ensemble Derivative of (Gl ul) sum l with respect to r is then

E(G) = dGl/drj uj ul   sum jl

Equating, sum ijkl on

dGl/drj uj ul = 1/J d/dvi [J dvi/drj drl/dvk Fk] uj ul

F and G are related by

Gi(r) ui = Fj(v) dri/dvj ui   sum j for any i

Both sides of E(G(r)) = E(F(v)) may be equated for fixed jl, since Octonion result product histories are identical. Looking closely at E(G(r))=E(F(v)), the chain rule may be written as

d/drj = 1/J d/dvi [J dvi/drj .....   sum i

This will be equivalent to the classic chain rule

d/drj = dvi/drj d/dvi   sum i

only if J and dvi/drj are constant in v. Perhaps the classic chain rule is not so general. Let me put the above form out as a better choice.

If we take the case of v=r a little further, we could define an Octonion differential operator D as

D = ui Di = ui d/dri   sum i

This differential operator multiplies like any other Octonion by the rules of the selected algebra representation. The partial differentiation is applied after algebraic multiplication as a scalar operation on the remaining product term components.

One very important thing to keep in mind is that for all diffeomorphisms to alternate coordinate system v, we never lose the fundamental basis units ui. They are always present and always define the operation of multiplication the same way for any choice of v. This implies Algebraic Invariance is coordinate system invariant.

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http://www.octospace.com/files/Octonion_Algebra_and_its_Connection_to_Physics.pdf