Now that we know there are two separate non-isomorphic Octonion Algebras; Left and Right types, and that each type may be expressed in eight isomorphic ways, it may seem we are closer to showing Octonion Algebras are too problematic for physics than we are to showing their virtues.

After all, any physical system will be described by functional expressions that are formed by one or more multiplication operations. We just saw that the definitions for Octonion multiplication vary between choice of representation, but we must insist that the mathematical results representing physically measurable entities are completely consistent.

The previous discussion on the variations allowed for the multiplication rules for Octonions did not indicate that any of the sixteen possible choices are special enough to forget about the others. In fact, they are all quite similar to each other. But all is not lost, since as we will soon see, it will be possible to form product terms that actually are consistent across all possible algebras. It would appear that all expected representations of observable phenomena will need to be consistent across all possible Octonion Algebra representations. I will indeed demand this and see where it leads. This is what I call "The Law of Octonion Algebraic Invariance".

Any time we form one or more product of Octonion quantities, we form a number of resultant product terms using the rules of our singular choice of Octonion Algebra. It must be a singular choice because you can not change the rules in the middle of the game. Since I have elected to keep consistent unit triplets within each permutation, replaying the entire product generation using a different form of Octonion Algebra will retain the same components in every product term. The only thing that might change is their sign.

A product term is an algebraic invariant if its sign does not change when the multiplication rules are changed to those of any other Octonion Algebra. A product term is an algebraic variant if its sign does change when the multiplication rules are changed to those of any other Octonion Algebra. Octonion product expressions may be full algebraic invariant, mixed variant and invariant, or fully variant.

When the variant product terms between two algebra choices are compared to those between two other choices, the intersection of the two sets may not be empty. Considering all possible changes, there is a way to separate the variant product terms into sets without intersection. I call these irreducible sets minimum distances. I have devised an algorithm to sift out invariant product terms and irreducible variant product terms into separate sets. I will not repeat its description here, it is available inside a PDF downloadable from my website. It is called "The Octonion Variance Sieve Process".

If an Octonion product expression is expected to be an algebraic invariant yet it is not explicitly invariant, it can be forced invariant by insisting each irreducible variant product set sums to zero. This is an assignment of algebraic equations of constraint, and may be very important to the application of Octonions to physics.

The question of algebraic variance and invariance is fully determined by the product history of a product term. Since we have lost associativity for multiplication, this history is registered in the order performed to reach the final product term configuration. For every fundamental multiplication operation, there is an effective unit number on both sides of the multiplication symbol. They may be straight up single units or a composite of earlier multiplications. Either way, it is the effective unit numbers that will determine the multiplication rule for that particular multiplication operation, and it will be the set of all applied multiplication rules that will define the algebraic variance properties of the product term.

It is convenient to have an invariant starting point for the eventual product history. If F = Fn un for n = 0 to 7, and Fn is a real number, a real valued function, or a differential operator then F is an algebraic invariant since there are no products of Octonion basis units in its definition.

The question of algebraic variance only manifests itself through the product of unlike vector basis units. The scalar u0 unit multiplies all other terms exactly the same for all Octonion Algebras. The same goes for like basis vector products. Neither of these product rules participate in the algebraic variance character of an Octonion product term.

The sign of the product of two unlike vector basis units is determined by the multiplication rules set within a single permutation. This single product is an algebraic variant, since there are a number of algebra changes that will negate its rule. If a second multiplication is performed that uses the same permutation rule, any negation from a change in algebra will be performed twice, leaving the final sign unchanged. This type of double product is an algebraic invariant. In fact, it is the only way to generate an algebraic invariant from a double product including unlike vector basis units. The even negation count is the mechanism by which algebraic invariants are created, although not necessarily requiring the same permutation rule. Invariant quad product terms can be formed using the rules of four separate permutations.

Any product term with an odd number of unlike vector basis unit products is an algebraic variant, since there will be a choice of algebra change that will produce an odd number of negations. This is not to say any Octonion triple product is an algebraic variant, for there are scalar and like basis unit products in the mix that will allow formation of invariant product terms.

Lets apply some of this to Octonion extensions to things we know from Electrodynamics. The 8-potential "A" and 8-differential operator "D" are assumed to be fundamental algebraic invariants, not formed by any Octonion product operations.

The decoupled 8-current density for Octonion unit [i] is

- uj uj DjDj(Ai) summed j = 0 to 7

This is an algebraic invariant, since the basis unit products are all like unit number. We would have expected this. Scalar charge density and vector current density must be algebraic invariants.

The irrotational field form for vector unit [i] is

ui u0 Di(A0) + u0 ui D0(Ai)

This field is an algebraic invariant, since it is a single product with one unit being scalar. This is a good thing, since we would demand the product of this field and the invariant scalar component of the 8-current yielding the central force form to also be invariant.

The rotational field form for vector unit [i] is

uj uk Dj(Ak) summation jk over (ijk)

This is an algebraic variant, since it is a single product defined by the multiplication rules of the single permutation (ijk). No big deal, since detectable force forms involve another vector unit multiplication. Lets look at this for the extended cross product of current density and rotational field. Since the 8-current is an algebraic invariant, we can simply call its kth component Jk. This force may then be written

Fj = Jk {Dj(Ak)-Dk(Aj)} uk(uj uk) ijk: (ijk)

This force is indeed an algebraic invariant, being the double application of the multiplication rules of (ijk).

Field contributions to energy density are straight forward algebraic invariants, like unit product only for irrotational fields, and like unit product with a double application of a single permutation rule for the rotational fields.

How about scalar work? It is the like unit product of two algebraic invariants, the irrotational field and the

8-current. Again, an algebraic invariant as expected.

Finally, look at the Octonion extension of the Poynting vector. It is the extended cross product of irrotational field and rotational field. Since the irrotational field is an algebraic invariant, the argument is identical to the cross product force above, just substitute the irrotational field for the 8-current. Again, Octonion Algebraic Invariance is there every time it needs to be.

I have fit Electrodynamics within Octonion Algebra from fields, to current density, to forces, to work-force action, to the analogy of the stress-energy-momentum tensor. Every Electrodynamic form needing to be an algebraic invariant actually is such.

So far, I have only shown that the multiplicity of definitions for the algebra of Octonions are not much of a problem for the description of physical phenomenon. All places where the variation would matter are shown to be algebraic invariants, consistent across all valid forms of the algebra. While this is certainly important for justifying the connection between Octonion Algebra and physics, it is only part of the story on Octonion Algebraic Invariance.

One of the best things we might hope for would be for our mathematical framework to provide a guiding light to greater understanding of the physical world. The brightest light would be mandated structure. We do not guess then verify, we are told how it must be up front. We have such with the concept of Octonion Algebraic Invariance. If we look inside Octonion Algebra for things we understand well like Electrodynamics, we find them as well as other invariant forms we may never know how to piece together without the insight provided by the full fundamental structure of Octonion Algebra and what it mandates algebraic invariant forms look like.

Algebraic Invariance principles can also be used to modify Octonion differential equations to other equivalent forms. The analogous Octonion stress-energy-momentum "tensor" formation requires changing the work-force action form, which is the algebraic invariant portion of the Octonion product of 8-current and field, to an integrable form which requires an outside differentiation on every product term. I tried the guess and try method on this for longer than I would like to admit. Then I got the bright idea to put my trust in the fundamental truth of Algebraic Invariance. Since the action expression was the full complement of invariance of an Octonion product, I looked for the full complement of algebraic invariant forms for the product string

ui [(uj uk) (ul um)]

Within part time days, I was able to find the identity. It is published in a PDF on my website and is laid out in a discussion thread to follow. The form has a factor of >16 more product terms than the work-force product of field and the sum of 8-current and 8-gradient of the Octonion equivalent of the Lorentz Condition, which as with standard classical Electrodynamics is present explicitly in the identity. This is what made it so difficult to do without the guiding light of Algebraic Invariance.

If one would run the associator: 1/2(a*(b*c)-(a*b)*c) through the Octonion Variance Sieve Process, they would find the portion sieved out as invariant is identically zero. This is so because as I mentioned, the only way to get an invariant two-product is a double application of the rules from a single permutation. This will require two of the three basis units involved to be identical. It is a fact that any Octonion subalgebra with only two basis units is associative. This does not imply all Octonion algebraically invariant forms are associative, they are not. However, it would appear the "concept" of Algebraic Invariance is associative. By this, I mean if one finds a string of unit products that is an algebraic invariant, any order changes created by insertion of parenthesis will also be an algebraic invariant. The permutation rule set may change, as might the sign of the product term, but it will be an algebraic invariant. In the interest of full disclosure, I have not proved this is always the case, I have only inspected a good number of cases. Some may ask, so what? I am just planting seeds, that is all.

In conclusion, the variability within the full definition of Octonion Algebra is one of its most important attributes. For when we try to reconcile it with the easily acceptable Law of Octonion Algebraic Invariance, we are given structure to exploit. Recalculation using a different selection for the applied algebra results in product terms that either change sign or do not. When all sixteen algebras are considered, algebraic invariants take on mandated form, which matches expectations when we have them, and shows the way when we may not have enough information. The algebraic variants can be sieved into irreducible sets that can provide overall Algebraic Invariance when their sums are individually assigned a value of zero. When we are dealing with differential forms, the irreducible sets provide homogeneous equations of algebraic constraint.

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For more information see
http://www.octospace.com/files/Octonion_Algebra_and_its_Connection_to_Physics.pdf