The multiplication rules for the seven non-scalar Octonion basis units are best described by seven permutations, any of which when combined with the scalar basis unit form a quaternion subalgebra.

I would like to use the short-hand notation (ijk) to imply the permutation basis unit multiplication rules for i,j,k not equal to each other and not equal to 0

```
ui * uj = uk    uj * uk = ui    uk * ui = uj
uj * ui = -uk   uk * uj = -ui   ui * uk = -uj
```

(ijk) == (jki) == (kij) equivalent cyclic permutations

Each of the seven permutations define six multiplication table entries, so 42 of 64 positions of the Octonion multiplication table are covered by permutation rules.

The variability in the set of possible Octonion algebras is entirely covered by the juxtaposition of units within the seven permutations, since for any algebra the remaining 22 basis unit products are singularly defined for all Octonion algebras for i not 0 as

```
ui * ui = -u0
u0 * ui = ui * u0 = ui
u0 * u0 = u0
```

Each ui (i not 0) appears in three of the seven permutations. The nature of a permutation triplet is such that the permutation multiplication rules above are negated by the exchange in position of any two members. Any second exchange restores the permutation to its original configuration.

The three negations of (ijk) are

```
(jik)  exchange ij
(kji)  exchange ik
(ikj)  exchange jk
```

The equivalent cyclic permutations for the negation are

(jik) == (kji) == (ikj)

If we do not allow unit aliasing, we must insist all attempts to modify Octonion Algebra do not alter the triplet of units in the set of permutations. That is to say if we started with (123) we do not end up with (124). We will only allow the negation of one or more permutations as a possible change to the multiplication rules. This will set the maximum number of different algebra definitions to 128. However, not all 128 of these moves result in valid Octonion Algebra.

For "/a" representing the conjugate of Octonion "a" and for "b" also Octonion we must have

/a * (a * b) = (/a * a) * b

If we check this equality against the 128 possible rules, we find only sixteen work, and therefore are valid Octonion rules. As will be shown, the sixteen fall into two sets of eight I call "Left Octonion Algebra" and "Right Octonion Algebra". The members of each type are all algebraically isomorphic, but Left is not algebraically isomorphic to Right.

Isomorphism is an overloaded term in mathematics. Perhaps it would be helpful to discuss what is meant by the phrase "algebraically isomorphic". Two algebras are isomorphic if and only if their multiplication tables are equivalent. "Equivalent" does not mean identical. A row pair may be exchanged followed by the exchange of same column pair, changing the appearance of the table without changing the product definition of any two basis units, the two tables are "equivalent".

The distinction between "Left" and "Right" Octonion Algebra is clearly seen within the three permutations that include any one of the seven basis units. Cyclically shift these permutations such that the common basis unit is in the central position. The three basis units on the right end of these three permutations will be members of a fourth permutation and those on the left end will not for "Right" Octonion Algebra. Similarly, the three basis units on the left end will be members of a fourth permutation and those on the right end will not for "Left" Octonion Algebra.

The partitioning between the three permutations that include one of the basis units and the four that do not is important for understanding the permutation negations that map within or between Left and Right Octonion Algebra types. To see this, let us examine the seven permutations assuming one is (ijk).

Within the three permutations including basis unit ui, basis units uj and uk are only present in permutation (ijk). Now consider the four permutations that do not contain ui. Two of four will include the basis unit uj and not uk, the other two will include the basis unit uk and not uj.

For instance, take (ijk) = (123) in the following Right Octonion Algebra

(312)   has 1,2,3
(415)   has 1
(617)   has 1

(257)   has 2
(264)   has 2
(365)   has 3
(374)   has 3

If we negate the four permutations that do not include ui, then negate the resultant permutations that do not include uj, the two permutations that include uk but not ui or uj will be negated twice therefore will remain unchanged. The permutation (ijk) is not negated at all. The two permutations that include ui but are not (ijk) are negated once, and the two without ui or uk are also negated once. The net result is that for a defined permutation (ijk), negating the four permutations that do not include ui followed by negating the four permutations that do not include uj is equivalent to negating the four permutations that do not include uk. Clearly this may be repeated as many times as one may like with the final result identical to a single negation of all permutations that do not include a single unit. If you have guessed the operation of negating the four permutations that do not include one of the seven vector basis units is an algebraic isomorphism, you would be correct as will be shown.

Now look at negating the three permutations that include basis unit ui followed by negating the three permutations that include basis unit uj. The two permutations that include ui that are not (ijk) are negated once. (ijk) is negated twice so remains unchanged. In the set of four permutations that do not include ui, the two with uj and not uk are negated once, and the two with uk and not uj are not negated at all. The net result once again is for defined permutation (ijk) negating the three permutations that include ui followed by negating the three permutations that include uj is equivalent to negating the four permutations that do not include uk.

To close out double negations, lets do three/four. Order does not matter here. After negation of the three permutations that include ui, negate the four permutations that do not include uj. The two permutations including ui that are not (ijk) are negated twice so remain unchanged. The two permutations that include uj but not ui or uk are not ever negated, and the two that include uk but not ui or uj are negated once, and (ijk) is negated once. The net result is that any four/without and three/with negation combination for ui and uj is equivalent to negating the three permutations that include uk.

If you have guessed the negation of the three permutations that include one of the seven basis units is not an algebraic isomorphism, you get a gold star. In fact, the two negation schemes above are the ONLY schemes that will result in a valid Octonion Algebra.

Any member of Left or Right Octonion Algebra is related to one of the seven others of the same type by the negation of all permutations that do not include one of the seven basis units, one to one. The Left/Right cross type negation scheme for each of the sixteen to one of eight of the other type is either the negation of all seven permutations, or the negation of the three permutations that include one of the seven basis units; one to one of seven.

For those who loathed word problems, some specific negation examples might help to visualize this.

original		not u1		not u2	
(123)		(123)		(123)	unchanged, has u3
(761)		(761)	->	(671)	changed, no u3
(572)	->	(752)		(752)	changed, no u3
(653)	->	(563)	->	(653)	unchanged, has u3
(541)		(541)	->	(451)	changed, no u3
(642)	->	(462)		(462)	changed, no u3
(743)	->	(473)	->	(743)	unchanged, has u3

original		with u1		with u2	
(123)	->	(213)	->	(123)	unchanged, has u3
(761)	->	(671)		(671)	changed, no u3
(572)		(572)	->	(752)	changed, no u3
(653)		(653)		(653)	unchanged, has u3
(541)	->	(451)		(451)	changed, no u3
(642)		(642)	->	(462)	changed, no u3
(743)		(743)		(743)	unchanged, has u3

original		with u1		not u2	
(123)	->	(213)		(213)	changed, has u3
(761)	->	(671)	->	(761)	unchanged, no u3
(572)		(572)		(572)	unchanged, no u3
(653)		(653)	->	(563)	changed, has u3
(541)	->	(451)	->	(541)	unchanged, no u3
(642)		(642)		(642)	unchanged, no u3
(743)		(743)	->	(473)	changed, has u3

original		not u1		with u2	
(123)		(123)	->	(213)	changed, has u3
(761)		(761)		(761)	unchanged, no u3
(572)	->	(752)	->	(572)	unchanged, no u3
(653)	->	(563)		(563)	changed, has u3
(541)		(541)		(541)	unchanged, no u3
(642)	->	(462)	->	(642)	unchanged, no u3
(743)	->	(473)		(473)	changed, has u3

Notice how the union of arrows on the top two and likewise on the bottom two fills in all positions.

Lets finish up on the algebraic isomorphism idea. Working out the four specific examples we just did, my claim is that the top two are algebraic isomorphisms, and the bottom two are not. The top two are equivalent to a single negation of all permutations that do not contain u3. If we gather up the three unchanged permutations with u3 central, pick any two permutations, exchange between the left two units and exchange between the right two units, the three are recovered:

Try exchanges 2 <-> 5, and 1 <-> 6

(231) -> (536)
(536) -> (231)
(437) -> (437)

Continuing the same exchanges on the remaining four:

(671) -> (176)
(752) -> (725)
(451) -> (426)
(462) -> (415)

These exchanges have restored the original permutation set. Exchanges carried out consistently on all seven permutations are nothing more than a consistent renaming of the units, so the difference between the two representations is basis unit naming convention only. The algebras are indeed isomorphic.

Now for the proposed non algebraic isomorphisms. Again, grab the three permutations containing u3, with u3 central:

original	final
(231)	(132)
(536)	(635)
(437)	(734)

Grab another three permutations with some other unit in common, say u5:

original	final
(257)	(257)
(653)	(356)
(154)	(154)

For the u3 common triples, 1, 6 and 7 remained on a common side, yet the side switched as required for the move from Right to Left Octonion Algebra. For the u5 common triples, the side for another permutation changed as expected, again from Right to Left, but the permutation changed from (734) to (231).

It might be sufficient to some to already see that the consistent side swap for one of the other permutations resulting from a change between Right and Left Octonion Algebras would unlikely be remapped back to the original side by a consistent name exchange for any combination of units. If more is needed, lets try the unit exchange method.

For u3 common, the exchange map back by unit name exchange is 2 <-> 1, 5 <-> 6, and 4 <-> 7. Carrying this out on the remaining four permutations:

(761) -> (452)
(572) -> (641)
(541) -> (672)

(642) -> (571)

For the u5 common, the exchange map back is simply 3 <-> 6. Carrying this out on the remaining four permutations:

(213) -> (216)
(761) -> (731)
(642) -> (342)
(473) -> (476)

In both cases, the remaining four permutations exchange to new triplets not found in the original. The unit exchange mechanism fails. The algebra formed by negating the three permutations which include any single unit is not isomorphic to the original algebra.

The three permutations that include one of the basis units can always be used to determine the unit naming map between any other isomorphic representation, even those with quite different triplet choices. There are many different mappings between any two. Take for instance the algebra created by starting with (124) and adding one modulo 8 omitting 0 to each unit to form the permutation set

(124)
(235)
(346)
(457)
(561)
(672)
(713)

First, identify if this is Right or Left Octonion by examining the three permutations that include your choice of unit number. I will pick u2. Arranging we have

(124)
(523)
(726)

Units u4, u3 and u6 are members of one of our seven permutations so we have Right Octonion Algebra. Notice none of the triplets match any of those in the Right algebra at the top of this discussion. If we grab any arbitrary unit in the first algebra, say unit u1, then place the three permutations including u1 in arbitrary order we have a one to one mapping between these two representations of Right Octonion Algebra.

(617) <-> (124)
(312) <-> (523)
(415) <-> (726)

This is equivalent to

6 <-> 1

1 <-> 2
7 <-> 4
3 <-> 5
2 <-> 3
4 <-> 7
5 <-> 6

Performing the left to right map on our original algebra

(123) -> (235)
(761) -> (412)
(572) -> (643) -> (346)
(653) -> (165) -> (561)
(541) -> (672)
(642) -> (173) -> (713)
(743) -> (475) -> (457)

The second change is the isomorphic map negating all permutations that do not include u2. This gets us to the second form of Right algebra.

It will be quite worthwhile to look closely at the isomorphic negation scheme. There are two triplets of units assignable to every vector basis unit. One triplet is a permutation and one is not. For instance, take unit u4 and arrange the seven Right algebra permutations as follows

```
(123) (761) (541)
      (572) (642)
      (653) (743)
```

(123) is the "fourth" permutation associated with basis unit u4. u5, u6 and u7 are the non permutation basis units associated with basis unit u4. The three permutations that include u4 give the one:one connections u1:u5, u2:u6 and u3:u7. Now make an association between these two triplets and our familiar rectangular physical x, y and z as follows

{xyz} : {123} : {567}

Permutation (123) covers {xyz} with a closed set rule for multiplication. {567} covers {xyz} with an open set rule for multiplication requiring three permutations that each include one unit of (123). Notice that (123) is an {xyz} right hand rule and {567} is an {xyz} left hand rule.

When the four permutations that do not include u4 are negated in the Octonion algebraic isomorphism, the {xyz} opposite hand relationship is preserved with (123) and {567} both changing. This is the case for both Right and Left Octonion Algebras. The unit exchange scheme that above was successful in recovering the original permutation unit orders above will be {5 <-> 6 and 1 <-> 2}, or {5 <-> 7 and 1 <-> 3} or {6 <-> 7 and 2 <-> 3} here with unit u4 instead of u3 used in the above example. Any of these exchanges can be seen to be an {xyz} hand rule change for both (123) and {567} if the {xyz} connection to both remains fixed. This effectively undoing the negation of the four permutations that do not include u4 through renaming.

The variability in {xyz} handedness is non-confrontational to the concept of Octonion algebraic isomorphism. The variability only arises when an extra-algebraic connection is fixed between algebraic units and non-algebraic {xyz}. These geometric assignments are of no fundamental concern to the algebra itself, even though we will need to make an association in order to apply Octonion Algebra to physical reality.

In conclusion on the subject of the full cover of the definition of Octonion Algebra, there are sixteen different juxtapositions of units within the seven permutations central to the definition of the algebra. The set of sixteen is actually two sets of eight, defining eight Left Octonion Algebras, and eight Right Octonion Algebras. All eight in each type are isomorphic algebras. However, Left Octonion Algebra is not isomorphic to Right Octonion Algebra.

The negation of the four permutations that do not include one of the Octonion basis units is an isomorphism of algebras. The negation of the three permutations that do include one of the Octonion basis units is not an isomorphism of algebras.

The only valid permutation negation schemes are the negation of all three permutations that include one of the Octonion basis units, the negation of all four permutations that do not include one of the Octonion basis units, or multiple combinations of these two. Any other scheme will not produce a valid Octonion Algebra. All combinations of valid negations will produce a representation found in the group of sixteen.

This last paragraph is extremely important to discussions to come when I will discuss the concept of algebraic invariance and variance.

What follows is an itemization of the 16 different permutation configurations for Left and Right Octonion Algebra. Taking column 0 as the prototype, column n not 0 of the same type is column 0 changed by negating the permutations that do not include basis unit n. The map between types is same column negation of the permutations that include the basis unit u4.

Left O Algebra

Column	0	1	2	3	4	5	6	7
	(123)	(123)	(123)	(123)	(321)	(321)	(321)	(321)
	(761)	(761)	(167)	(167)	(167)	(167)	(761)	(761)
	(572)	(275)	(572)	(275)	(275)	(572)	(275)	(572)
	(653)	(356)	(356)	(653)	(356)	(653)	(653)	(356)
	(145)	(145)	(541)	(541)	(145)	(145)	(541)	(541)

	(246)	(642)	(246)	(642)	(246)	(642)	(246)	(642)
	(347)	(743)	(743)	(347)	(347)	(743)	(743)	(347)

Right O Algebra

Column	0	1	2	3	4	5	6	7
	(123)	(123)	(123)	(123)	(321)	(321)	(321)	(321)
	(761)	(761)	(167)	(167)	(167)	(167)	(761)	(761)
	(572)	(275)	(572)	(275)	(275)	(572)	(275)	(572)
	(653)	(356)	(356)	(653)	(356)	(653)	(653)	(356)
	(541)	(541)	(145)	(145)	(541)	(541)	(145)	(145)
	(642)	(246)	(642)	(246)	(642)	(246)	(642)	(246)
	(743)	(347)	(347)	(743)	(743)	(347)	(347)	(743)

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