Quaternion Doubling for Right and Left Octonion Algebra

In keeping with my current enumeration scheme for the sixteen Right and Left Octonion Algebras[1], we start with a prototype Right form R0, and construct Rj for j not 0 as the algebra defined by negation of the four permutation multiplication rules of R0 that do not include e_j. The Left Algebra Lj is the anti-automorphism of Rj, that is the negation of all seven permutation rules.

Their enumerations are as follows.

Right **O** Algebra

R0	R1	R2	R3	R4	R5	R6	R7
$(e_1\ e_2\ e_3)$	$(e_1\ e_2\ e_3)$	$(e_1\ e_2\ e_3)$	$(e_1\ e_2\ e_3)$	$(e_3\ e_2\ e_1)$	$(e_3\ e_2\ e_1)$	$(e_3\ e_2\ e_1)$	$(e_3\ e_2\ e_1)$
$(e_7\ e_6\ e_1)$	$(e_7\ e_6\ e_1)$	$(e_1\ e_6\ e_7)$	$(e_1\ e_6\ e_7)$	$(e_1\ e_6\ e_7)$	$(e_1\ e_6\ e_7)$	$(e_7\ e_6\ e_1)$	$(e_7\ e_6\ e_1)$
$(e_5\ e_7\ e_2)$	$(e_2\ e_7\ e_5)$	$(e_5\ e_7\ e_2)$	$(e_2\ e_7\ e_5)$	$(e_2\ e_7\ e_5)$	$(e_5\ e_7\ e_2)$	$(e_2\ e_7\ e_5)$	$(e_5\ e_7\ e_2)$
$(e_6\ e_5\ e_3)$	$(e_3\ e_5\ e_6)$	$(e_3\ e_5\ e_6)$	$(e_6\ e_5\ e_3)$	$(e_3\ e_5\ e_6)$	$(e_6\ e_5\ e_3)$	$(e_6\ e_5\ e_3)$	$(e_3\ e_5\ e_6)$
$(e_5\ e_4\ e_1)$	$(e_5\ e_4\ e_1)$	$(e_1\ e_4\ e_5)$	$(e_1\ e_4\ e_5)$	$(e_5\ e_4\ e_1)$	$(e_5\ e_4\ e_1)$	$(e_1\ e_4\ e_5)$	$(e_1\ e_4\ e_5)$
$(e_6\ e_4\ e_2)$	$(e_2\ e_4\ e_6)$	$(e_6\ e_4\ e_2)$	$(e_2\ e_4\ e_6)$	$(e_6\ e_4\ e_2)$	$(e_2\ e_4\ e_6)$	$(e_6\ e_4\ e_2)$	$(e_2\ e_4\ e_6)$
$(e_7\ e_4\ e_3)$	$(e_3\ e_4\ e_7)$	$(e_3\ e_4\ e_7)$	$(e_7\ e_4\ e_3)$	$(e_7\ e_4\ e_3)$	$(e_3\ e_4\ e_7)$	$(e_3\ e_4\ e_7)$	$(e_7\ e_4\ e_3)$

Left **O** Algebra

L0	L1	L2	L3	L4	L5	L6	L7
$(e_3\ e_2\ e_1)$	$(e_3\ e_2\ e_1)$	$(e_3\ e_2\ e_1)$	$(e_3\ e_2\ e_1)$	$(e_1\ e_2\ e_3)$	$(e_1\ e_2\ e_3)$	$(e_1\ e_2\ e_3)$	$(e_1\ e_2\ e_3)$
$(e_1\ e_6\ e_7)$	$(e_1\ e_6\ e_7)$	$(e_7\ e_6\ e_1)$	$(e_7\ e_6\ e_1)$	$(e_7\ e_6\ e_1)$	$(e_7\ e_6\ e_1)$	$(e_1\ e_6\ e_7)$	$(e_1\ e_6\ e_7)$
$(e_2\ e_7\ e_5)$	$(e_5\ e_7\ e_2)$	$(e_2\ e_7\ e_5)$	$(e_5\ e_7\ e_2)$	$(e_5\ e_7\ e_2)$	$(e_2\ e_7\ e_5)$	$(e_5\ e_7\ e_2)$	$(e_2\ e_7\ e_5)$
$(e_3\ e_5\ e_6)$	$(e_6\ e_5\ e_3)$	$(e_6\ e_5\ e_3)$	$(e_3\ e_5\ e_6)$	$(e_6\ e_5\ e_3)$	$(e_3\ e_5\ e_6)$	$(e_3\ e_5\ e_6)$	$(e_6\ e_5\ e_3)$
$(e_1\ e_4\ e_5)$	$(e_1\ e_4\ e_5)$	$(e_5\ e_4\ e_1)$	$(e_5\ e_4\ e_1)$	$(e_1\ e_4\ e_5)$	$(e_1\ e_4\ e_5)$	$(e_5\ e_4\ e_1)$	$(e_5\ e_4\ e_1)$
$(e_2\ e_4\ e_6)$	$(e_6\ e_4\ e_2)$	$(e_2\ e_4\ e_6)$	$(e_6\ e_4\ e_2)$	$(e_2\ e_4\ e_6)$	$(e_6\ e_4\ e_2)$	$(e_2\ e_4\ e_6)$	$(e_6\ e_4\ e_2)$
$(e_3\ e_4\ e_7)$	$(e_7\ e_4\ e_3)$	$(e_7\ e_4\ e_3)$	$(e_3\ e_4\ e_7)$	$(e_3\ e_4\ e_7)$	$(e_7\ e_4\ e_3)$	$(e_7\ e_4\ e_3)$	$(e_3\ e_4\ e_7)$

It is well known that a doubling process on the Quaternions can produce the multiplication table for an Octonion Algebra. This may be demonstrated by starting with a particular Quaternion Algebra form, choosing a new anti-commutating square root of -1 basis element, and then multiplying the four Quaternion basis elements by this new element to produce four more basis elements for a total of eight.

There are some free choices on just how to do this. There are two isomorphic yet different appearing multiplication rules for the non-scalar Quaternion elements, represented say by the two permutation multiplication rules $(e_1\ e_2\ e_3)$ and $(e_3\ e_2\ e_1)$. We also understand that there will be four additional non-scalar Octonion basis elements, and any one of them will do as a doubler on the Quaternions. We must also understand that the doubling multiplication may be performed either from the left side or the right side of the given Quaternion set.

As I have previously demonstrated [2][3], all Octonion Algebras demonstrate a handedness where Left Octonion Algebras all indicate the three permutation rules including any single basis element will have the elements of an additional permutation set cyclically to the left of the common basis element, and Right Octonion will have an additional permutation set cyclically to the right of the common basis element. Since all seven permutations will exhibit this rule with a particular basis element in any valid Octonion Algebra, it would be correct to assume the side on which we multiply the doubling element will correspond to the handedness of the resultant Octonion Algebra. It also follows that the doubling element must be multiplied from the same side on each of the original Quaternion non-scalar basis elements, else the result will not be a valid Octonion Algebra.

If we start with (e_1 e_2 e_3) and choose e_4 as the new element on which to define the additional four basis elements for a Left Octonion Algebra, the new elements will be the set {e_4, (e_1 * e_4), (e_2 * e_4), (e_3 * e_4)}. We have already established our seven basis element triplets on which to form the permutation rules, so we will have (e_1 * e_4) = e_5, (e_2 * e_4) = e_6, (e_3 * e_4) = e_7. We then have for Left Octonion doubling from (e_1 e_2 e_3) with e_4 the four permutation rules

(e_1 e_2 e_3)
(e_1 e_4 e_5)
(e_2 e_4 e_6)
(e_3 e_4 e_7)

We must now determine the multiplication rules for (e_5 * e_6), (e_6 * e_7), and (e_7 * e_5).

Knowing we must abide by the handedness rule for Left Octonion Algebras and that we have already definitions for two permutations each including either e_1, e_2, or e_3, we may gather up the three permutations that include one of them at a time to pick up the three yet unknown permutation rules in turn. Starting with e_1

(e_3 e_1 e_2)
(e_5 e_1 e_4)
(e_a e_1 e_b)

Since we have the triplet {e_3 e_6 e_5} and the algebra is Left Octonion, index a must be 6 therefore index b is 7, so we add permutation (e_6 e_1 e_7) to the mix. Next gather permutations with e_2 and apply the same, this gives

(e_1 e_2 e_3)
(e_6 e_2 e_4)   yielding
(e_7 e_2 e_5)

And finally gather permutations with e_3 and apply the same, this gives

(e_2 e_3 e_1)
(e_7 e_3 e_4)   yielding

$(e_5\ e_3\ e_6)$

Our seven permutations from Left Algebra doubling with e_4 on $(e_1\ e_2\ e_3)$ are uniquely

$(e_1\ e_2\ e_3)$
$(e_6\ e_1\ e_7)$
$(e_7\ e_2\ e_5)$
$(e_5\ e_3\ e_6)$
$(e_1\ e_4\ e_5)$
$(e_2\ e_4\ e_6)$
$(e_3\ e_4\ e_7)$

This is seen to be Left Octonion Algebra L4.

Next do Left Algebra doubling with e_4 on $(e_3\ e_2\ e_1)$. We have

$(e_3\ e_2\ e_1)$
$(e_1\ e_4\ e_5)$
$(e_2\ e_4\ e_6)$
$(e_3\ e_4\ e_7)$

Then

$(e_2\ e_1\ e_3)$
$(e_5\ e_1\ e_4)$ yielding
$(e_7\ e_1\ e_6)$

$(e_3\ e_2\ e_1)$
$(e_6\ e_2\ e_4)$ yielding
$(e_5\ e_2\ e_7)$

$(e_1\ e_3\ e_2)$
$(e_7\ e_3\ e_4)$ yielding
$(e_6\ e_3\ e_5)$

Now our seven permutations from Left Algebra doubling with e_4 on $(e_3\ e_2\ e_1)$ are uniquely

$(e_3\ e_2\ e_1)$
$(e_7\ e_1\ e_6)$
$(e_5\ e_2\ e_7)$
$(e_6\ e_3\ e_5)$
$(e_1\ e_4\ e_5)$
$(e_2\ e_4\ e_6)$
$(e_3\ e_4\ e_7)$

This is recognized as Left Octonion Algebra L0.

In a similar fashion now do Right Algebra doubling with e_4 on (e_1 e_2 e_3). We have the four permutations

(e_1 e_2 e_3)
(e_5 e_4 e_1)
(e_6 e_4 e_2)
(e_7 e_4 e_3)

(e_3 e_1 e_2)
(e_4 e_1 e_5) yielding
(e_6 e_1 e_7)

(e_1 e_2 e_3)
(e_4 e_2 e_6) yielding
(e_7 e_2 e_5)

(e_2 e_3 e_1)
(e_4 e_3 e_7) yielding
(e_5 e_3 e_6)

Our seven permutations from Right Algebra doubling with e_4 on (e_1 e_2 e_3) are uniquely

(e_1 e_2 e_3)
(e_6 e_1 e_7)
(e_7 e_2 e_5)
(e_5 e_3 e_6)
(e_5 e_4 e_1)
(e_6 e_4 e_2)
(e_7 e_4 e_3)

This is seen to be Right Octonion Algebra R0.

Similarly do Right Algebra doubling with e_4 on (e_3 e_2 e_1)

(e_3 e_2 e_1)
(e_5 e_4 e_1)
(e_6 e_4 e_2)
(e_7 e_4 e_3)

(e_2 e_1 e_3)
(e_4 e_1 e_5) yielding
(e_7 e_1 e_6)

(e_3 e_2 e_1)
(e_4 e_2 e_6) yielding

(e_5 e_2 e_7)

(e_1 e_3 e_2)
(e_4 e_3 e_7) yielding
(e_6 e_3 e_5)

The seven permutations from Right Algebra doubling with e_4 on (e_3 e_2 e_1) are uniquely

(e_3 e_2 e_1)
(e_7 e_1 e_6)
(e_5 e_2 e_7)
(e_6 e_3 e_5)
(e_5 e_4 e_1)
(e_6 e_4 e_2)
(e_7 e_4 e_3)

This is seen to be Right Octonion Algebra R4.

We could chunk through doubling (e_1 e_2 e_3) and (e_3 e_2 e_1) with e_5, e_6, and e_7 in the same way, but it would be easier to understand that once a doubling triplet, doubling basis element and multiplication side are chosen, the resultant Octonion Algebra is determined, since the initial four permutations are unique to a single algebra.

Summarizing the results, we have the full cover of all sixteen Octonion Algebras

Right doubling (e_1 e_2 e_3) with e_4	R0
Right doubling (e_3 e_2 e_1) with e_4	R4
Left doubling (e_1 e_2 e_3) with e_4	L4
Left doubling (e_3 e_2 e_1) with e_4	L0
Right doubling (e_1 e_2 e_3) with e_5	R3
Right doubling (e_3 e_2 e_1) with e_5	R6
Left doubling (e_1 e_2 e_3) with e_5	L6
Left doubling (e_3 e_2 e_1) with e_5	L3
Right doubling (e_1 e_2 e_3) with e_6	R1
Right doubling (e_3 e_2 e_1) with e_6	R7
Left doubling (e_1 e_2 e_3) with e_6	L7
Left doubling (e_3 e_2 e_1) with e_6	L1
Right doubling (e_1 e_2 e_3) with e_7	R2
Right doubling (e_3 e_2 e_1) with e_7	R5
Left doubling (e_1 e_2 e_3) with e_7	L5
Left doubling (e_3 e_2 e_1) with e_7	L2

© Richard Lockyer, May 12, 2010

It is interesting to note the index juxtapositions we have in this summary. The original choice for a prototype algebra R0 is a doubled form from (e_1 e_2 e_3) and e_4, and we have in the doubled from e_4 group e_4, R0, R4, L0 and L4. In the next group we have e_5, R3, R6, L3 and L6 and {e_5 e_3 e_6} is a permutation triplet. Then in the next group we have e_6, R1, R7, L1 and L7 and {e_6 e_1 e_7} is a permutation triplet. In the last group we have e_7, R2, R5, L2 and L5 and {e_7 e_2 e_5} is a permutation triplet. These three permutation triplets are what remains from the full set of seven after removing the doubling Quaternion set {e_1 e_2 e_3} and the three not including e_4, the basis element used to double for R0. This comes about from the index definition scheme for the algebras being based on the basis elements. Although the indexing choice is arbitrary, the particular choice here points out that there is like structure within structure provided by the Octonions, we just need to look for it and make particular choices amongst seemingly arbitrary ones to emphasize what is ubiquitous. There are more examples of like structure within structure here [1].

Now we could have started with any of the other six Octonion permutation rules as our basis for the initial Quaternion Algebra. Then the chosen permutation triplet and the four unassigned basis elements could be put through the above process with similar results but with different combinations for the cover of the sixteen Octonion Algebras.

Bibliography

[1] (2010) R. Lockyer, “Hadamard Matrix Connection to Octonion Algebras”
http://www.octospace.com/files/Hadamard_Matrix_Connection_to_Octonion_Algebra.pdf

[2] (2008) R. Lockyer, “Octonion Algebra and its Connection to Physics”
http://www.octospace.com/files/Octonion_Algebra_and_its_Connection_to_Physics.pdf

[3] sci.physics post, R. Lockyer, “Why Octonions 2: A Full Description of the Algebra”
http://groups.google.com/group/sci.physics/browse_thread/thread/8e026683e7eb1149/53c1e9e00f4d5fe1?q
August 8, 2008