On Isomorphism Within The Octonions

The Definition of an n Dimensional Algebra [1]

An n dimensional algebra is a set of expressions of the form

(1) $a_1 \, i_1 + a_2 \, i_2 + \ldots + a_n \, i_n$

where $a_1, a_2, \ldots a_n$ are arbitrary real numbers, and $i_1, i_2, \ldots i_n$ are objects called basis elements with the following operations:

1. Multiplication by a real number

   (2) $k \, (a_1 \, i_1 + a_2 \, i_2 + \ldots + a_n \, i_n) = ka_1 \, i_1 + ka_2 \, i_2 + \ldots + ka_n \, i_n$

2. Addition

   (3) $(a_1 \, i_1 + a_2 \, i_2 + \ldots + a_n \, i_n) + (b_1 \, i_1 + b_2 \, i_2 + \ldots + b_n \, i_n)$
   $=$
   $(a_1 + b_1) \, i_1 + (a_2 + b_2) \, i_2 + \ldots + (a_n + b_n) \, i_n$

3. Multiplication given in terms of a table of products

   (4) $i_r \, i_s = p_{rs,1} \, i_1 + p_{rs,2} \, i_2 + \ldots + p_{rs,n} \, i_n$

   where r and s are integers 1 to n. This table of products is used to determine the product

   $(a_1 \, i_1 + a_2 \, i_2 + \ldots + a_n \, i_n) \; (b_1 \, i_1 + b_2 \, i_2 + \ldots + b_n \, i_n)$

Operations 1 and 2 demonstrate no opportunity for distinction between algebras of the same dimension. An n dimensional algebra is therefore completely determined by operation 3, specifically by its multiplication table (4), that is to say by the n^3 terms $p_{rs,t}$.

It is informative to investigate the impact on the definition of multiplication when the basis element set is modified. In (4) above, replace i_m with $k_m \, i_m$ where k_m are any non-zero real numbers. We then have as the definition for multiplication relative to this new and *different* basis:

(5) $(k_r \, i_r) \, (k_s \, i_s) = p_{rs,1} \, (k_1 \, i_1) + p_{rs,2} \, (k_2 \, i_2) + \ldots + p_{rs,n} \, (k_n \, i_n)$

To demonstrate the next point, it is beneficial to disguise the connection to the original basis i_m with a new and equivalent representation $j_m = k_m i_m$. Then (5) becomes

(6) $j_r \; j_s \; = \; p_{rs,1} j_1 + p_{rs,2} j_2 + \ldots + p_{rs,n} j_n$

If we compare the multiplication table (6) with multiplication table (4), we would find that j_m occurs everywhere i_m does, in the same juxtaposition with every p term. Although i_m is different from j_m the rule of multiplication is applied equally to basis i_m as it is to j_m. The multiplication tables and hence the algebras defined by (4) and (6) are identical by virtue of both having identical $p_{rs,t}$. This is called an isomorphism of algebras. This concept is more generally defined as

Definition of Algebraic Isomorphism[1]

"Two n-dimensional algebras are said to be isomorphic if they have bases with identical multiplication tables"

An important takeaway from all of this is that it is not the basis that defines the rules for multiplication, it is the set of coefficients found in the results for all possible products of two basis elements, the $p_{rs,t}$ above. The algebra is not defined *by the basis,* it is defined *on the basis.* Within a given algebra, the rules for multiplication are applied equivalently to *all basis choices.* This is well known by the physicist, since quite often basis changes are made to make real world problems more readily solvable. It is important to have the freedom to do so without concern that the consistency of the fundamental mathematics, its algebra, is compromised in any way.

It is also important to recognize the fact that (4) and (6) are not simply algebraic expressions when it comes to the concept of isomorphism. The i_m and j_m are *generalized representations* of particular basis choices within the algebra *defined by* the n^3 terms $p_{rs,t}$. The values for k_m in $j_m = k_m i_m$ are chosen first, then *that representation* is inserted into the expression for the definition of multiplication and therefore the algebra itself. The result of a multiplication is then expressed in terms of coefficients on the basis elements *as the basis elements were originally defined*.

To this point, it is quite improper to assume a modification from (4) to (4') such that a sign change to say, all i_4 *in any way* changes the definition of the prescribed algebra, even though *algebraically* we can say we will not modify (4') by associating this negation not with the basis element 4, but instead with all p terms touched by this element, and all element product sides that include one basis element i_4. This algebraic manipulation on (4') can be considered a two step operation. First, the basis terms are modified by un-negating all i_4. This, as shown above, is an isomorphism. The next move required to stay algebraically consistent with (4') is negating all p where one element on the element product side is i_4, and all $p_{rs,4}$. This *is not* an isomorphism since the p's are no longer what they started out as. If we call (4') algebraically modified in this way (4''), what we have actually demonstrated is that the algebra prescribed by (4) *is* isomorphic with the algebra prescribed by (4'), but *is not* isomorphic to the algebra prescribed by (4'').

I have shown[2][3] that there are two separate and distinct multiplication tables for octonion algebra, called Left and Right algebras. In the same references I have shown that the method to produce one type from the other is the negation of all three permutations that include any one of the basis elements. I have called them non-isomorphic since there is no re-enumeration of basis elements for one that will yield a match between their respective multiplication table coefficients $p_{rs,t}$.

The only way the Left and Right algebras could be called isomorphic is to "hack" the definition by claiming isomorphism is demonstrated if one can find a particular basis choice in the first algebra that yields an expression for its multiplication table that is algebraically equivalent to that for a particular choice of basis in the other algebra. You must further ignore the fact that for whatever basis representation you made in the first algebra, basis representations can be found in the other that *do not* yield algebraically equivalent expressions for their respective multiplication tables.

The fact that you can demonstrate the "hack" of algebraically equivalent multiplication table expressions between Left and Right octonion algebras by negating an arbitrary *odd* number of non-scalar basis elements in one is indeed interesting. It does provide a certain degree of sameness between them, but this is not complete enough in my humble opinion to be an isomorphism, since any *even* number of non-scalar basis element negations in one retains algebraically different expressions for their respective multiplication tables. This *is* while it *is not* seems quite Clintonesque, where apparently "it depends on what your definition of the word is is". Once you realize that you *actually did* have sex with that algebra when you moved the minus signs off the basis elements to the $p_{rs,t}$, the contradictory statement goes away. Left is always Left, and Right is always Right.

The concept of isomorphism is not central to any of the concepts I have presented as application of octonion algebra to physical reality. Indeed, my Octonion Variance Sieve Process[2] exploits rule differences not only between Left and Right octonion representations, but also within each of the eight isomorphic ways to represent Left algebras and eight isomorphic ways to represent Right algebras. I leave the rarified air of abstract mathematical thought behind when I strive to attach extra-algebraic physical significance to each of the octonion basis elements. This physical attachment must be singular in deference to our singular physical reality. When a single enumerated basis choice is applied within all possible multiplication rule sets for the octonions, it becomes natural to evaluate how individual product terms change when the multiplication rules are changed, hence the sieve process. Some product terms will not change between applications of any two of the sixteen Left and Right multiplication rules, they are algebraic invariants. The Law of Algebraic Invariance[2] states that non-zero octonion expressions for physical reality must be formed with only algebraically invariant products. A corollary to this is the sieved out sets of algebraically variant product terms sum individually to zero to allow them to be implicit invariants since +zero = -zero. These are algebraic equations of constraint.

Bibliography

(1)

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(2)

http://www.octospace.com/files/Octonion_Algebra_and_its_Connection_to_Physics.pdf

(3)

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