For the purposes of this paper, take as the Right Octonion Algebra[1] prototype R(0) the definition expressed in the permutation triplet set

(e_1 e_2 e_3) (e_7 e_6 e_1) (e_5 e_7 e_2) (e_6 e_5 e_3) (e_5 e_4 e_1) (e_6 e_4 e_2) (e_7 e_4 e_3)

If we look at just the signs of the products in the multiplication table, we have the following

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	+1	+1	+1	+1	+1	+1	+1	+1
e_1	+1	-1	+1	-1	-1	+1	-1	+1
e_2	+1	-1	-1	+1	-1	+1	+1	-1
e_3	+1	+1	-1	-1	-1	-1	+1	+1
e_4	+1	+1	+1	+1	-1	-1	-1	-1
e_5	+1	-1	-1	+1	+1	-1	-1	+1
e_6	+1	+1	-1	-1	+1	+1	-1	-1
e_7	+1	-1	+1	-1	+1	-1	+1	-1

Vectors formed from the rows of this matrix are orthogonal since they have an inner product <a,b> = 0. The same holds for the columns. The inner product of any row or column with itself <a,a> = 8, the dimension of the vectors. This makes the matrix what is called a Hadamard matrix of dimension 8. This connection is well known[2].

This matrix may be placed in a symmetric form by reordering the rows.

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	+1	+1	+1	+1	+1	+1	+1	+1
e_6	+1	+1	-1	-1	+1	+1	-1	-1
e_7	+1	-1	+1	-1	+1	-1	+1	-1
e_5	+1	-1	-1	+1	+1	-1	-1	+1
e_4	+1	+1	+1	+1	-1	-1	-1	-1
e_3	+1	+1	-1	-1	-1	-1	+1	+1
e_1	+1	-1	+1	-1	-1	+1	-1	+1
e_2	+1	-1	-1	+1	-1	+1	+1	-1

Now the matrix of signs is of the form

A	A
A	-A

where A is the 4x4 matrix

+1	+1	+1	+1
+1	+1	-1	-1
+1	-1	+1	-1
+1	-1	-1	+1

Define the Left Octonion Algebra[1] prototype L(0) as the algebra where all permutation rules for R(0) are negated. This algebra may be considered the commutation of the Right Octonion Algebra R(0), since changing the order of multiplication will be the same as negating the rules for all permutations. The multiplication table signs for the Left Octonion Algebra prototype L(0) are members of the table

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	+1	+1	+1	+1	+1	+1	+1	+1
e_1	+1	-1	-1	+1	+1	-1	+1	-1
e_2	+1	+1	-1	-1	+1	-1	-1	+1
e_3	+1	-1	+1	-1	+1	+1	-1	-1
e_4	+1	-1	-1	-1	-1	+1	+1	+1
e_5	+1	+1	+1	-1	-1	-1	+1	-1
e_6	+1	-1	+1	+1	-1	-1	-1	+1
e_7	+1	+1	-1	+1	-1	+1	-1	-1

The symmetric form for L(0) is a reorder on the columns instead of the rows just as a Right(Left) Octonion multiplication table may be formed from a Left(Right) Octonion table by transposing rows and columns.

	e_0	e_6	e_7	e_5	e_4	e_3	e_1	e_2
e_0	+1	+1	+1	+1	+1	+1	+1	+1
e_1	+1	+1	-1	-1	+1	+1	-1	-1
e_2	+1	-1	+1	-1	+1	-1	+1	-1
e_3	+1	-1	-1	+1	+1	-1	-1	+1
e_4	+1	+1	+1	+1	-1	-1	-1	-1
e_5	+1	+1	-1	-1	-1	-1	+1	+1
e_6	+1	-1	+1	-1	-1	+1	-1	+1
e_7	+1	-1	-1	+1	-1	+1	+1	-1

There is a one to one connection between individual basis elements and permutations for each of the eight Right or Left Octonion Algebras that will be the same for R(i) and L(i) but different between R(i) and R(j), or L(i) and L(j) for i not equal to j. They may be found by taking the non-scalar basis elements one at a time, cyclically rotating the three permutation triplets the chosen basis element belongs to until the chosen basis element is in the central position. For Right Octonion, the three basis elements to the right of the chosen element is the permutation triplet associated with it, as will the one on the left for Left Octonion. This is also the defining characteristic of “Right” and “Left” Octonion Algebras, and the method to determine which you are dealing with.

The associations are as follows for R(0) and L(0) respectively

(e_1 e_2 e_3) and (e_1 e_3 e_2)	e_4
(e_7 e_6 e_1) and (e_7 e_1 e_6)	e_3
(e_5 e_7 e_2) and (e_5 e_2 e_7)	e_1
(e_6 e_5 e_3) and (e_6 e_3 e_5)	e_2
(e_5 e_4 e_1) and (e_5 e_1 e_4)	e_6
(e_6 e_4 e_2) and (e_6 e_2 e_4)	e_7
(e_7 e_4 e_3) and (e_7 e_3 e_4)	e_5

We will use these permutations as components of the algebras and use their associations to the individual basis elements as a replacement scheme in the symmetric Hadamard sign table.

Define a representation of our choice for the Right Octonion Algebra prototype R(0) with the "vector" RP. Define the components of RP as follows indexed with their associated basis element indices:

RP(0) == the set of basis products not covered by permutation triplet rules.
RP(6) == the closed set of basis product rules of the permutation triplet (e_5 e_4 e_1)
RP(7) == the closed set of basis product rules of the permutation triplet (e_6 e_4 e_2)
RP(5) == the closed set of basis product rules of the permutation triplet (e_7 e_4 e_3)
RP(4) == the closed set of basis product rules of the permutation triplet (e_1 e_2 e_3)
RP(3) == the closed set of basis product rules of the permutation triplet (e_7 e_6 e_1)
RP(1) == the closed set of basis product rules of the permutation triplet (e_5 e_7 e_2)
RP(2) == the closed set of basis product rules of the permutation triplet (e_6 e_5 e_3)

Then, we may define the full set of product term rules for the Right Octonion prototype R(0) as the inner product <RP, I>, where I is the identity {+1, +1, +1, +1, +1, +1, +1, +1} and RP = { RP(0), RP(6), RP(7), RP(5), RP(4), RP(3), RP(1), RP(2) }.

We may similarly define the Dual of RP for Left Octonion to be LP with components

LP(0) == the set of basis products not covered by permutation triplet rules.
LP(6) == the closed set of basis product rules of the permutation triplet (e_5 e_1 e_4)
LP(7) == the closed set of basis product rules of the permutation triplet (e_6 e_2 e_4)
LP(5) == the closed set of basis product rules of the permutation triplet (e_7 e_3 e_4)
LP(4) == the closed set of basis product rules of the permutation triplet (e_1 e_3 e_2)
LP(3) == the closed set of basis product rules of the permutation triplet (e_7 e_1 e_6)
LP(1) == the closed set of basis product rules of the permutation triplet (e_5 e_2 e_7)
LP(2) == the closed set of basis product rules of the permutation triplet (e_6 e_3 e_5)

LP = { LP(0), LP(6), LP(7), LP(5), LP(4), LP(3), LP(1), LP(2) }.

Of course RP(0) = LP(0). These sets are invariant to definition changes between the sixteen possible Octonion Algebras since $e_0 * e_i$ and $e_i * e_i$ are consistently defined across all sixteen Octonion Algebras.

The rule sets for all eight Right Octonion Algebras can be formed from linear combinations of the RP(n) elements, where the coefficients are either +1 or -1. The following table replaces the e_i in the symmetric $e_i * e_j$ sign table row label with their associated algebra component set RP(i). The symmetric table column labels e_j are replaced with Right Octonion R(j). The Hadamard table columns now correctly specify the linear combinations of RP(n) for each of the Right Octonion Algebras.

	R(0)	R(1)	R(2)	R(3)	R(4)	R(5)	R(6)	R(7)	permutation
RP(0)	+1	+1	+1	+1	+1	+1	+1	+1	none
RP(6)	+1	+1	-1	-1	+1	+1	-1	-1	(e_5 e_4 e_1)
RP(7)	+1	-1	+1	-1	+1	-1	+1	-1	(e_6 e_4 e_2)
RP(5)	+1	-1	-1	+1	+1	-1	-1	+1	(e_7 e_4 e_3)
RP(4)	+1	+1	+1	+1	-1	-1	-1	-1	(e_1 e_2 e_3)
RP(3)	+1	+1	-1	-1	-1	-1	+1	+1	(e_7 e_6 e_1)
RP(1)	+1	-1	+1	-1	-1	+1	-1	+1	(e_5 e_7 e_2)
RP(2)	+1	-1	-1	+1	-1	+1	+1	-1	(e_6 e_5 e_3)

Likewise, the eight Left Octonion Algebras can be formed from linear combinations of the LP(n), where the table of coefficients is identical to that for RP(n) combinations since we associate the negated permutations with LP.

	L(0)	L(1)	L(2)	L(3)	L(4)	L(5)	L(6)	L(7)	permutation
LP(0)	+1	+1	+1	+1	+1	+1	+1	+1	none
LP(6)	+1	+1	-1	-1	+1	+1	-1	-1	(e_5 e_1 e_4)
LP(7)	+1	-1	+1	-1	+1	-1	+1	-1	(e_6 e_2 e_4)
LP(5)	+1	-1	-1	+1	+1	-1	-1	+1	(e_7 e_3 e_4)
LP(4)	+1	+1	+1	+1	-1	-1	-1	-1	(e_1 e_3 e_2)
LP(3)	+1	+1	-1	-1	-1	-1	+1	+1	(e_7 e_1 e_6)
LP(1)	+1	-1	+1	-1	-1	+1	-1	+1	(e_5 e_2 e_7)
LP(2)	+1	-1	-1	+1	-1	+1	+1	-1	(e_6 e_3 e_5)

Both of these representations of Right and Left Octonion Algebras may be seen to specify what I have called the "elemental moves"[1][3] between algebras of the same type. Algebra R(j) for j not zero is Algebra R(0) with all four permutations not including e_j negated, and the same for L(j) except we start with L(0). The columns then "operate" on the multiplication rules in order to change the overall definition of the particular algebra.

Define the operator Iso(i) == H(, i)[5] as the i^{th} column vector of the symmetric Hadamard sign matrix H. Then we may compactly define a representation of all Right and Left Octonion Algebras as

R(i) = < Iso(i), RP >

L(i) = < Iso(i), LP >

The inner product here should be interpreted as the sum of rules individually operated on by the operator Iso().

We may define a composition of Iso() operators where: {do C then do B} is identical to do the single operation {A} defines A in terms of B and C as

A(n) = B(n)C(n) for n = 0-7

or equivalently, define

A = B · C

This composition definition on the operator Iso() is a commutative and associative algebra.

Iso() may be interpreted as an operator that morphs the partitioned components of the algebra R(0) or L(0) into one of the others of the same type. We may represent this as

R(j) = Iso(j) ⊗ R(0)
L(j) = Iso(j) ⊗ L(0)

Clearly Iso(0) is the identity morph. The operation for Iso(j) for j not 0 may be described as negating the four permutations that do not include the basis element e_j .

We can use the composition rule for Iso() on the last two expressions.

Iso(i) ⊗ R(j) = [Iso(i) · Iso(j)] ⊗ R(0)
Iso(i) ⊗ L(j) = [Iso(i) · Iso(j)] ⊗ L(0)

Examining the various combinations of Iso(i) · Iso(j) we find

Iso(i) · Iso(i) = Iso(0)
Iso(0) · Iso(j) = Iso(j)
Iso(i) · Iso(j) = Iso(k) for {ijk} one of the octonion permutation triplet index set.

From these rules Iso(i) can be shown to morph between any Right Algebras R(j) and R(k) or between any Left Algebras L(j) and L(k). It is not possible to use Iso(i) to map between Left and Right Octonion Algebras. The elemental map between Right and Left requires the negation of the three permutations that all include one of the basis elements. The sign vector for this move would have five +1 and three -1, and as such cannot be represented by a Hadamard matrix row or column. It can however be represented by a vector of sign components.

Define Dual(0) == {+1, -1, -1, -1, -1, -1, -1, -1}[5]

From our definitions of Right Octonion Algebras R(i) and Left Octonion Algebras L(i), we may write

$R(j) = Dual(0) \otimes L(j)$
$L(j) = Dual(0) \otimes R(j)$

We may also define Dual(j) == Dual(0) · Iso(j)

From this we can see Dual(i) also forms a commutative and associative algebra with itself and Iso(j). The compositions may be shown to be

$Dual(i) \cdot Dual(i) = Iso(0)$
$Dual(0) \cdot Dual(j) = Iso(j)$
$Dual(i) \cdot Dual(j) = Iso(k)$ for {ijk} one of the octonion permutation triplet index set.

$Dual(i) \cdot Iso(i) = Dual(0)$
$Dual(0) \cdot Iso(j) = Dual(j)$
$Dual(j) \cdot Iso(0) = Dual(j)$
$Dual(i) \cdot Iso(j) = Dual(k)$ for {ijk} one of the octonion permutation triplet index set.

We then have the following

$R(i) = Iso(0) \otimes R(i)$
$R(i) = Iso(i) \otimes R(0)$
$R(0) = Iso(i) \otimes R(i)$
$R(i) = Iso(j) \otimes R(k)$ for {ijk} a valid permutation triplet of basis indices

$L(i) = Iso(0) \otimes L(i)$
$L(i) = Iso(i) \otimes L(0)$
$L(0) = Iso(i) \otimes L(i)$
$L(i) = Iso(j) \otimes L(k)$ for {ijk} a valid permutation triplet of basis indices

$L(i) = Dual(0) \otimes R(i)$
$L(i) = Dual(i) \otimes R(0)$
$L(0) = Dual(i) \otimes R(i)$
$L(i) = Dual(j) \otimes R(k)$ for {ijk} a valid permutation triplet of basis indices

$R(i) = Dual(0) \otimes L(i)$
$R(i) = Dual(i) \otimes L(0)$
$R(0) = Dual(i) \otimes L(i)$
$R(i) = Dual(j) \otimes L(k)$ for {ijk} a valid permutation triplet of basis indices

The final connection is my Octonion Variance Sieve Process[1][4]. This process sorts out product terms from any octonion expression into invariant and variant sets. An octonion expression must be evaluated using a singular choice of Right or Left Octonion Algebra. The invariant set includes all product terms that will not change sign when the expression is evaluated in any of the other Octonion Algebras. The variant sets are product terms that will all change signs together when the algebra is changed to any other. The variant sets have no intersection. The two invariant sets are identical.

Define the expression result using Right Octonion Algebra R(j) as RR(j), and the expression result using Left Octonion Algebra L(j) as LR(j). The Left(Right) sieves are linear combinations of adds or subtracts of results over all Left(Right) expression results. The signs correspond to the rows of our symmetric Hadamard table when RR(j) and LR(j) are labels in the same columns as R(j) and L(j).

The invariant Left Octonion set is IL and the invariant Right Octonion set is IR. These are 1/8 the straight sums of Right Results for IR and Left Results for IL. These are the signs of the top row of the symmetric Hadamard matrix. Not coincidentally, this is the row labeled by the algebra rules independent of the choice of algebra, RP(0) and LP(0).

I will depart slightly from my previous definitions for the variant sets by not changing the order of the associated permutation index triplets between Right and Left sieves, and do my now customary avoidance of implied order by encompassing the three indices with {}. The signs for the sum/difference of Right Results RR(j) and Left Results LR(j) for 8x the sieves sr{lmn} and sl{lmn} respectively are found in the row where permutation triplet {lmn} is associated with RP(i) and LP(i) labeling the rows above.

The sieve symmetric table is then

Left sieve		LR(0)	LR(1)	LR(2)	LR(3)	LR(4)	LR(5)	LR(6)	LR(7)
	Right sieve	RR(0)	RR(1)	RR(2)	RR(3)	RR(4)	RR(5)	RR(6)	RR(7)
IL	IR	+1	+1	+1	+1	+1	+1	+1	+1
sl{541}	sr{541}	+1	+1	-1	-1	+1	+1	-1	-1
sl{642}	sr{642}	+1	-1	+1	-1	+1	-1	+1	-1
sl{743}	sr{743}	+1	-1	-1	+1	+1	-1	-1	+1
sl{123}	sr{123}	+1	+1	+1	+1	-1	-1	-1	-1
sl{761}	sr{761}	+1	+1	-1	-1	-1	-1	+1	+1
sl{572}	sr{572}	+1	-1	+1	-1	-1	+1	-1	+1
sl{653}	sr{653}	+1	-1	-1	+1	-1	+1	+1	-1

The sieved sets are[1]

IR	IL
½ [sl{541} + sr{541}]	½ [sl{541} - sr{541}]
½ [sl{642} + sr{642}]	½ [sl{642} - sr{642}]
½ [sl{743} + sr{743}]	½ [sl{743} - sr{743}]
½ [sl{123} + sr{123}]	½ [sl{123} - sr{123}]
½ [sl{761} + sr{761}]	½ [sl{761} - sr{761}]
½ [sl{572} + sr{572}]	½ [sl{572} - sr{572}]
½ [sl{653} + sr{653}]	½ [sl{653} - sr{653}]

All of the above representations on the symmetric Hadamard matrix are combined in the following summary table.

row	row	column	column	table	table								
L sieve						LR(0)	LR(1)	LR(2)	LR(3)	LR(4)	LR(5)	LR(6)	LR(7)
	R sieve					RR(0)	RR(1)	RR(2)	RR(3)	RR(4)	RR(5)	RR(6)	RR(7)
		L **O**				L(0)	L(1)	L(2)	L(3)	L(4)	L(5)	L(6)	L(7)
			R **O**			R(0)	R(1)	R(2)	R(3)	R(4)	R(5)	R(6)	R(7)
				L(0)		e_0	e_6	e_7	e_5	e_4	e_3	e_1	e_2
					R(0)	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
IL	IR	LP(0)	RP(0)	e_0	e_0	+1	+1	+1	+1	+1	+1	+1	+1
sl{541}	sr{541}	LP(6)	RP(6)	e_1	e_6	+1	+1	-1	-1	+1	+1	-1	-1
sl{642}	sr{642}	LP(7)	RP(7)	e_2	e_7	+1	-1	+1	-1	+1	-1	+1	-1
sl{743}	sr{743}	LP(5)	RP(5)	e_3	e_5	+1	-1	-1	+1	+1	-1	-1	+1
sl{123}	sr{123}	LP(4)	RP(4)	e_4	e_4	+1	+1	+1	+1	-1	-1	-1	-1
sl{761}	sr{761}	LP(3)	RP(3)	e_5	e_3	+1	+1	-1	-1	-1	-1	+1	+1
sl{572}	sr{572}	LP(1)	RP(1)	e_6	e_1	+1	-1	+1	-1	-1	+1	-1	+1
sl{653}	sr{653}	LP(2)	RP(2)	e_7	e_2	+1	-1	-1	+1	-1	+1	+1	-1

To summarize, the Hadamard sign matrix connects Right and Left Octonion Algebra prototypes to the full set of sixteen Right and Left forms. It further connects up the Octonion Variance Sieve Process to the full set of Octonion Algebras.

Prototype R(0) $\rightarrow$ R **O** $\rightarrow$ R Sieve
$\downarrow$
Prototype L(0) $\rightarrow$ L **O** $\rightarrow$ L Sieve

It is interesting to note that the set of Right Octonion Algebras and the operation Iso(), as well as the set of Left Octonion Algebras and the operation Iso(), form proper Abelian Groups[6]. The construction of the non-commutative, non-associative Right and Left Octonion Algebras *define* these group characteristics. The group characteristics *do not* define the algebras. I am hopeful that the application of group theoretical considerations on this clean connection to Octonions will yield new insights that will *only* be realized on return to the core foundations, the algebra itself.

I would like to thank Jens Koeplinger for his insights and desire to refine, clarify and extend ideas I have tried to promote about the Octonions. His idea to abstract the Octonion multiplication table morphs[7] was instrumental in the above crystallization of what I had previously presented in perhaps overly wordy and difficult to understand terms. I wish him happy hunting on his desire to extend our understanding of physical reality through considerations on the Octonions.

Bibliography

[1] (2008) R. Lockyer, "Octonion Algebra and its Connection to Physics"
http://www.octospace.com/files/Octonion_Algebra_and_its_Connection_to_Physics.pdf

[2] (1994) Geoffrey Dixon, Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics, Kluwer Academic Publishers

[3] sci.physics post, R. Lockyer, "Why Octonions 2: A Full Description of the Algebra"
http://groups.google.com/group/sci.physics/browse_thread/thread/8e026683e7eb1149/53c1e9e00f4d5fe1?q
August 8, 2008

[4] sci.physics post, R. Lockyer "Why Octonions 3: Octonion Algebraic Variance and Invariance"
http://groups.google.com/group/sci.physics/browse_thread/thread/f13f82ef48fe78c4
August 8, 2008

[5] hypercomplex tech group post, R. Lockyer
http://tech.groups.yahoo.com/group/hypercomplex/message/824

[6] hypercomplex tech group post, R. Lockyer
http://tech.groups.yahoo.com/group/hypercomplex/message/832

[7] hypercomplex tech group post, Jens Koeplinger
http://tech.groups.yahoo.com/group/hypercomplex/message/821